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MINIMAL NIL-TRANSFORMATIONS OF CLASS TWO

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ABSTRACT

On a metric minimal flow (X, a) which is a torus (K) extension of its largest almost periodic factor $Z = X/K$, the following conditions are equivalent.

(i) (X, a) is a nil-transformation of the form $(N/\Gamma, a)$ where K is central in N and $[N, N] \subset K$.

(ii) $E(X)$, the enveloping group of (X, a) is a nilpotent group of class 2.

(iii) Any minimal subset Ω of $X \times X$ is invariant under the diagonal action of K and the quotient $\Omega/K = Z_1$, is the largest almost periodic factor of Ω.

The enveloping groups of such flows are described and a corollary on cocycles of the circle into itself is deduced. Finally general minimal nil-transformations of class two are shown to be of the form considered in condition (i) above (possibly with a different nilpotent group) and consequently we deduce that the class of minimal flows which are group factors of nil-transformations of class 2 is closed under factors.

§1. Introduction

In [F,2] H. Furstenberg identifies ergodic nil-transformations T of class two as the "characteristic family" for ergodic sums of the form $\frac{1}{N}\sum_{n=0}^{N-1} T^n f\ T^{2n} g\ T^{3n} h$. In a forthcoming work, with B. Weiss, they show that in order to study these ergodic sums for a general ergodic easure preserving transformation, it is enough to consider factors of the form $(X, \mathcal{B}, \mu, T)$, where $X = Z \times_\varphi K$, Z the largest Kronecker factor of X, K a compact metric abelian group, φ a measurable cocycle of Z into K and $T(z,k) = (z + \tau, \varphi(z)k)$, ($\tau$ is a generator for the compact

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monothetic group Z). Further analyzing these characteristic factors they show that a certain functional equation, Lesigne's equation, is satisfied. The final step in their analysis is to deduce from this equation the existence of a nilpotent topological group N of class 2 such that the characteristic factor $(X, \mathcal{B}, \mu, T)$ is measure theoretically isomorphic to the nil-transformation $(N/\Gamma, a)$ where Γ is a closed subgroup and $a \in N$.

In [L], E. Lesigne shows that these ergodic sums in fact converge for such nil-transformations. On his way to prove this Lesigne shows, without explicitly stating it, that for a nil-transformation (X, a) of class 2 and any pair of points $x_1, x_2 \in X$, the orbit closure of (x_1, x_2) in $X \times X$ is isomorphic to a sub-flow of the form $X \times K$ where K is a group rotation. This led Professor Furstenberg to conjecture that this last property actually topologically characterizes nil-transformations of class 2. In this paper we prove a slightly restricted version of this conjecture and show that a third equivalent condition is that $E(X)$, the enveloping group of the flow (X, a) is (as an abstract group) a nilpotent group of class 2.

There are very few known explicit representations of enveloping semigroups (see [F,1], [N]). As a by-product of our main theorem we obtain an explicit representation for the enveloping groups of nil-transformations of class 2. (This includes as a special case Namioka's computation). We get as a special case of the main theorem a dynamical characterization of those functions φ, from the circle into itself, which with respect to an irrational rotation, are co-homologous to a character. Another consequence of the main theorem is that the class of minimal nilflows of order two and their group factors is closed under passage to factors.

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§2. Definitions and Statement of Results

Let X be a compact metric space, $\mathcal{H}(X)$ denotes the group of self-homeomorphisms of X endowed with the topology of uniform convergence of homeomorphisms and their inverses. With this topology $\mathcal{H}(X)$ is a polish topological group.

Let a be an element of $\mathcal{H}(X)$ such that the corresponding transformation acts minimally on X. We shall call the couple (X,a) a minimal flow. Let $K \subset \mathcal{H}(X)$ be a compact commutative subgroup commuting with a. Thus each $k \in K$ is an automorphism of (X,a). Since such an automorphism has a fixed point if and only if it is the identity automorphism, we see that K acts freely on X. We further assume that the quotient map $X \xrightarrow{\pi} Z \cong X/K$ is the homomorphism of (X,a) onto its largest almost periodic factor. Choosing a point $z_0 \in Z$ we can consider Z as a compact monothetic topological group with identity element z_0. Fixing a point $x_0 \in X$ with $\pi(x_0) = z_0$ we find that $\pi(ax_0) = \tau \in Z$ is a generator for Z.

Our assumptions on the minimal flow (X,a) clearly imply that it is distal and we recall that the enveloping semigroup $E = E(X,a)$ of a distal flow is in fact a group. If $(\tilde{Z},\tilde{\tau})$ is the largest almost periodic factor of the flow (E,a) then the factor map $E \xrightarrow{\tilde{\pi}} \tilde{Z}$ is a group homomorphism and we let $E' = \ker\tilde{\pi}$. (See [E] page 135 for the definition and properties of the group E').

For elements g,h of a group G we write $[g,h] = ghg^{-1}h^{-1}$ and let $[G,G]$ be the subgroup generated by all elements of the form $[g,h]$. G is nilpotent of class 2 if $[G,G]$ is contained in the center of G.

For $x,x' \in X$, $\bar{o}(x,x')$ denotes the orbit closure of $(x,x') \in X \times X$ under $a \times a$. We are now ready to state our main result.

2.1 THEOREM: *Suppose K is a torus (finite or infinite dimensional). The following conditions on (X,a) are equivalent:*

- $\hat{1}$ *There exists a closed nilpotent group of class 2, $N \subset \mathcal{H}(X)$ acting transitively on X, with $a \in N$, $K \subset N$, K central in N and $[N,N] \subset K$, and a co-compact closed subgroup Γ of N such that the nil-flow $(N/\Gamma,a)$ is isomorphic to (X,a).*
- $\hat{2}$ *E (as an abstract group) is nilpotent of class 2.*
- $\hat{3}$ *$K = E'$.*
- $\hat{4}$ *For every $x_1 \in X$ the subgroup*

$$\Delta_K = \{(k,k) : k \in K\}$$

of $K \times K$ acts on $\Omega = \bar{o}(x_0,x_1)$ and the quotient map $\Omega \xrightarrow{\pi_1} \Omega/\Delta_K = Z_1$ is the largest almost periodic factor of Ω.

When these conditions are satisfied Γ is isomorphic to a subgroup of the group $\mathrm{Hom}_c(Z,K)$ of continuous homomorphisms of Z into K. If in addition $\hat{K}$, the *dual group of K, is finitely generated then Γ is a countable discrete subgroup of N and N itself is locally compact and σ-compact.*

In the proof of Theorem 2.1 the assumption that K is a torus is used only at one point (Claim 6.3); what we actually use is the following assumption: For every commutative compact topological group G, closed subgroup $H \subset G$ and a continuous homomorphism $\varphi : H \to K$ there exists a continuous homomorphism $\psi : G \to K$ such that $\psi \restriction H = \varphi$. However this is equivalent to K being a torus. In the general case we introduce condition

$\hat{1}^*$ *There exists a closed nilpotent group of class 2, $N \subset calH(X)$ acting transitively on X, with $a \in N, K \subset N, K$ central in N and $[N,N] \subset K$, a closed co-compact subgroup Γ of N and a compact commutative subgroup $W \subset N$ satisfying $wa = aw, \forall w \in W$ and $W \cap K = \{e\}$, such that the flow $(W \backslash N/\Gamma, a)$ (i.e. the quotient of $(N/\Gamma, a)$ under the group of automorphisms W) is isomorphic to (X, a).*

2.1* THEOREM: *Conditions $\hat{1}^*, \hat{2}, \hat{3}$ and $\hat{4}$ on the flow (X, a) are equivalent.*

We don't know of an example where condition $\hat{1}^*$ occurs but (X, a) is not isomorphic to a nil-flow. See however Example 6.4.

On the way of proving Theorem 2.1 we get the following result. For $X = N/\Gamma$ satisfying condition $\hat{1}$ of Theorem 2.1 let $x_0 = \Gamma \in X$, let $\varphi_0 : N \to K$ be defined by $\varphi_0(g) = [a, g]$ and let $\mathrm{Hom}(N, K)$ be the group of all (not necessarily continuous) homomorphisms of N into K equipped with its (compact) pointwise convergence topology. Clearly $\varphi_0 \in \mathrm{Hom}(N, K)$ and we let

$$\Phi = \text{ closure } \{\varphi_0^n : n \in \mathbb{Z}\}.$$

2.2 THEOREM: *Let $\tilde{E} =$ closure $\{(a^n x_0, \varphi_0^n) \in X \times \Phi : n \in \mathbb{Z}\}$. Then the formulas*

$$(g\Gamma, \varphi)(h\Gamma, \psi) = (\varphi(h)hg\Gamma, \varphi\psi),$$
$$(g\Gamma, \varphi)^{-1} = (\varphi(g)g^{-1}\Gamma, \varphi^{-1})$$

for $(g\Gamma, \varphi), (h\Gamma, \psi) \in \tilde{E}$, define a group structure on $\tilde{E}$. Multiplication on the left by $\tilde{a} = (a\Gamma, \varphi_0)$ is continuous and $(\tilde{E}, \tilde{a})$ is isomorphic as a flow and also as a group to (E, a).

Specializing to flows on the 2-torus $C \times C = X$ where $C = \{e^{2\pi i\theta} : \theta \in \mathbb{R}\}$, let α be an irrational number, $\tau = e^{2\pi i\alpha}$, and let $\varphi : C \to C$ be a continuous map.

Define

$$a : C \times C \to C \times C$$

by $a(z,w) = (\tau z, \varphi(z)w)$.

2.3 THEOREM: *Suppose (X,a) is minimal and that the projection $(X,a) \xrightarrow{\pi} (\mathbb{C},\tau)$ onto the first coordinate is the largest almost periodic factor. Then φ is cohomologous to one of the functions $k_0\varphi_n$: $\mathbb{C} \to \mathbb{C}$, $\varphi_n(z) = z^n$ ($n \in \mathbb{Z}\backslash\{0\}$, $k_0 \in C$) iff the flow (X,a) satisfies the conditions of Theorem 2.1.*

The general minimal nil-flow $(N/\Gamma, a)$ need not satisfy the condition $[N,N] \subset K$ (see Example 6.5). However we have

2.4 THEOREM: *Let (X,a) be a minimal metric flow, $N \subset \mathcal{H}(X)$ a closed subgroup, nilpotent of class 2 with $a \in N$, which acts transitively on X. Then*

1. *closure $[N,N] = H$ is a compact central subgroup of N.*
2. *There exists a compact central subgroup $K \subset H$ such that $(X/K, a) = (Z,\tau)$ is the largest almost periodic factor of (X,a).*
3. *There exists a closed subgroup $N_0 \subset N$ with, $a \in N_0$, $K \subset N_0$, $[N_0,N_0] \subset K$ and N_0 acts transitively on X; i.e. (X,a) satisfies condition $\hat{1}$ of Theorem 2.1.*

From this together with Theorem 2.1* we deduce

2.5 THEOREM: *The class of minimal flows of the form $(X,a) = (W\backslash N/\Gamma, a)$ where N is a nilpotent locally compact group of class 2, Γ a closed co-compact subgroup and W a compact abelian subgroup commuting with a is closed under factors.*

In [P], W.A. Parry proves a much stronger theorem for nil-flows $(N/\Gamma, a)$ of all classes. However, he assumes that N is connected. When N is a connected separable nilpotent locally compact group any compact subgroup W is central. It therefore follows that for such groups $W\backslash N/\Gamma = N/\Gamma W$.

In Section 3 we prove two key lemmas. In Section 4, Theorem 2.2 is proved as well as the implication $\hat{1} \Longrightarrow \hat{2}$. In Section 5 first the implications $\hat{2} \Longrightarrow \hat{3} \Longrightarrow \hat{4}$ are proved. The implication $\hat{4} \Longrightarrow \hat{1}$ turns out to be more delicate and we introduce an intermediary condition $\hat{4}^*$ which augments $\hat{4}$ and allows us to use Lemma 3.1 and prove $\hat{4}^* \Longrightarrow \hat{1}$.

In Section 6 we come back to assumption $\hat{4}$ and show that it implies the existence of a group extension (X^*, a^*) of (X,a) which satisfies condition $\hat{4}^*$

and is therefore (by Section 4) a nil-flow. This completes the proof of Theorem 2.1*. However to complete the proof of $\hat{4} \Longrightarrow \hat{1}$, we need the further assumption that K is a torus. Using this assumption we are able to show that also (X, a) is a nil-flow. In Section 7 Theorem 2.3 is proved, and in the last section we prove Theorems 2.4 and 2.5.

We use the notations $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $C = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ for the circle group as convenient.

§3. Two Lemmas

3.1 LEMMA: *Suppose (X, a) satisfies condition $\hat{4}$ of Theorem 2.1 and suppose further that for every $x_1 \in X$ the following conditions are satisfied:*

(i) *The subgroup $K_0 = \{k \in K : (e \times k)\Omega = \Omega\}$ is monothetic*

(ii) *There exists a topological generator $k_0 \in K_0$ and a flow homomorphism $\theta : (\Omega, a \times a) \to (K_0, k_0)$ with $\theta(x_0, x_1) = e$.*

(iii) *$\theta(kx, kx') = \theta(x, x')$ and $\theta(x, \ell x') = \ell\theta(x, x')$ for all $k \in K$, $\ell \in K_0$ and $(x, x') \in \Omega$.*

(iv) *The map $\eta : (\Omega, a \times a) \to (X \times K_0, a \times k_0)$, $\eta(x, x') = (x, \theta(x, x'))$ is an isomorphism of Ω onto the product flow $X \times K_0$.*

Then (X, a) satisfies condition $\hat{1}$ of Theorem 2.1.

We first prove the following

3.2 CLAIM: *There exists a homeomorphism $b \in \mathcal{H}(X)$ such that*

1. *$bx_0 = x_1$,*
2. *$ab = k_0 ba$,*
3. *$bk = kb$ for all $k \in K$,*
4. *if $(x, x') \in \Omega$ then $bx = \theta(x, x')^{-1}x'$.*

Proof: Since η is 1-1 there exists for each $x \in X$ a unique point $bx \in X$ for which $\eta(x, bx) = (x, e)$. It is easy to check that b is continuous. We have $\eta(x_0, x_1) = (x_0, \theta(x_0, x_1)) = (x_0, e)$ hence $bx_0 = x_1$. Since for each $x \in X$

$$\begin{aligned}\eta(ax, abx) &= (ax, \theta(ax, abx)) = (ax, k_0\theta(x, bx)) \\ &= (ax, k_0) = (e \times k_0)(ax, e) = (e \times k_0)\eta(ax, bax) \\ &= \eta(ax, k_0bax)\ ,\end{aligned}$$

we deduce that $ab = k_0ba$. For all $k \in K$ and $x \in X$, $\eta(kx, bkx) = (kx, e)$ and

$$\eta(kx, kbx) = (kx, \theta(kx, kbx)) = (kx, \theta(x, bx)) = (kx, e) .$$

Hence $bk = kb$.

For the proof of 4. Choose a sequence $n_i \in \mathbb{Z}$ such that $\lim a^{n_i} \times a^{n_i}(x_0, x_1) = (x, x') \in \Omega$. Then by continuity of b and compactness of K

$$x' = \lim a^{n_i} x_1 = \lim a^{n_i} bx_0 = \lim k_0^{n_i} ba^{n_i} x_0 = kbx$$

where $k = \lim k_0^{n_i}$ in K. Also

$$\begin{aligned}(x, \theta(x, x')) &= \eta(x, x') = \eta(\lim(a^{n_i}x_0, a^{n_i}x_1)) = \lim(a^{n_i}x_0, \theta(a^{n_i}x_0, a^{n_i}x_1)) \\ &= \lim(a^{n_i}x_0, k_0^{n_i}\theta(x_0, x_1)) = (x, k)\end{aligned}$$

and $\theta(x, x') = k$. Thus $kbx = x'$ implies $bx = \theta(x, x')^{-1}x'$ as claimed. The fact that b is onto follows since for every $x' \in X$ there exists an $x \in X$ with $(x, x') \in \Omega$. Also if $bx = b\bar{x}$ then clearly $\bar{x} = kx$ for some $k \in K$ and therefore $bx = b\bar{x} = bkx = kbx$. Since the action of K is free we have $k = e$ and $\bar{x} = x$. Thus b is a homeomorphism and the proof of Claim 3.2 is complete. ∎

Notice that we actually proved a "local" theorem: For the existence of $b \in \mathcal{H}(X)$ with $ab = k_0ba$ and $bk = kb$ $\forall k \in K$ such that $bx_0 = x_1$, all we need are the assumptions on the orbit closure $\Omega = \bar{o}(x_0, x_1)$, we don't use the "global" assumption that these conditions hold for every $x_1 \in X$.

We now let

$$N = \{b \in \mathcal{H}(X) : \exists k_0 \in K,\ ab = k_0ba \ \&\ \forall k \in K,\ bk = kb\}$$

3.3 Claim: *N is a closed subgroup of $\mathcal{H}(X)$ and $[N, N] \subset K$; in particular N is nilpotent of class 2.*

Proof: Clearly N is closed and $K \subset N$. If $b_i \in N$ with $ab_i = k_ib_ia$ $(i = 1, 2)$ then one easily checks that $ab_1b_2 = k_1k_2b_1b_2a$ and $ab_1^{-1} = k_1^{-1}b_1^{-1}a$, so that N is a group. If we let $h = [b_1, b_2]$ then

$$ah = ab_1b_2b_1^{-1}b_2^{-1} = k_1k_2k_1^{-1}k_2^{-1}ha = ha .$$

Now every element of N defines an automorphism of the flow (Z, τ). In particular h as a commutator defines the identity automorphism on Z; hence there exists

$k \in K$ for which $hx_0 = kx_0$. Since as we have seen $ha = ah$ this implies $h = k$ and $[N, N] \subset K$. Since K is central, N is nilpotent of class 2. ∎

By Claim 3.2 the action of N on X is transitive: let $\Gamma = \{\gamma \in N : \gamma x_0 = x_0\}$, then the natural map $N/\Gamma \to X$ is one to one and continuous.

In order to complete the proof of Lemma 3.1 we now need to show that Γ is co-compact in N, for then it will follow that this map is a homeomorphism. This will be done in the next Lemma. ∎

3.4 LEMMA: *Suppose there exists a closed nilpotent group $N \subset \mathcal{H}(X)$ acting transitively on X, with $a \in N, K \subset N, K$ central in N and $[N, N] \subset K$, then:*

1. *There is a homomorphism $f : N \to Z$ satisfying the equation*

$$\pi(bx) = f(b)\pi(x) \quad (x \in X,\ b \in N).$$

2. *The subgroup $\Gamma = \{\gamma \in N : \gamma x_0 = x_0\}$ is isomorphic to a subgroup of $\mathrm{Hom}_c(Z, K)$.*
3. *Γ is co-compact.*
4. *If $\hat{K}$, the dual group of K, is finitely generated then Γ is countable and discrete in N and N itself is locally compact and σ-compact.*

Proof:

1. Each element $b \in N$, since it commutes with K, defines a homeomorphism $\tilde{b} = F(b) \in \mathcal{H}(Z)$ and clearly the map $F : N \to \mathcal{H}(Z)$ is a group homomorphism with $K \subset \ker F$; in particular $[N, N] \subset \ker F$. We have $\tilde{a} = F(a) = \tau$ and for each $b \in N$

$$\tau\tilde{b} = \tilde{a}\tilde{b} = \widetilde{ab} = F([a, b]ba) = \tilde{b}\tilde{a} = \tilde{b}\tau .$$

Thus each $\tilde{b}$ is an automorphism of the flow (Z, τ) and is therefore a translation of Z by the element $\tilde{b}(z_0) = z_b$. We write $z_b = f(b)$ and $f : N \to Z$ is then a group homomorphism satisfying

$$\pi(bx) = f(b)\pi(x) \qquad (x \in X,\ b \in N) .$$

2. Since for $\gamma \in \Gamma$, $\tilde{\gamma}(z_0) = f(\gamma) = z_0$ we have $\tilde{\gamma}(z) = z$ for every $z \in Z$. Thus for every $x \in X$, $\pi(\gamma x) = \pi(x)$ and we conclude that there exists a continuous map $\hat{\gamma} : X \to K$ with $\gamma x = \hat{\gamma}(x)x$. For $k \in K$, $x \in X$ we have

$\hat{\gamma}(kx)kx = \gamma kx = k\gamma x = k\hat{\gamma}(x)x$. Hence $\hat{\gamma}(kx) = \hat{\gamma}(x)$ and $\hat{\gamma}$ is defined on $X/K = Z$. Also

$$\begin{aligned}\gamma(ax) &= \hat{\gamma}(ax)ax \\ &= [\gamma, a]a\gamma x = [\gamma, a]a\hat{\gamma}(x)x = k_\gamma\hat{\gamma}(x)ax\end{aligned}$$

where we put $[\gamma, a] = k_\gamma \in K$. Hence $\hat{\gamma}(ax) = k_\gamma\hat{\gamma}(x)$ and for every $n \in \mathbb{Z}$ also $\hat{\gamma}(a^n x) = k_\gamma^n\hat{\gamma}(x)$. On Z we have

$$\begin{aligned}\hat{\gamma}(\pi(a^n x)) = \hat{\gamma}(\pi(a^n x_0)\pi(x)) = \hat{\gamma}(\tau^n\pi(x)) = k_\gamma^n\ \hat{\gamma}(\pi(x)) \\ = \hat{\gamma}(\tau^n)\hat{\gamma}(\pi x)\ .\end{aligned}$$

By continuity we get $\hat{\gamma}(zz') = \hat{\gamma}(z)\hat{\gamma}(z')$ for all $z, z' \in Z$, and $\hat{\gamma}$ is a continuous homomorphism. For $\gamma_1, \gamma_2 \in \Gamma$ we have

$$\gamma_2\gamma_1(x) = \widehat{\gamma_2\gamma_1}(x)x = \gamma_2(\hat{\gamma}_1(x)x) = \hat{\gamma}_2(x)\hat{\gamma}_1(x)x$$

so that $\widehat{\gamma_2\gamma_1} = \hat{\gamma}_2 \cdot \hat{\gamma}_1$. Similarly $\widehat{\gamma^{-1}} = \hat{\gamma}^{-1}$ and $\gamma \to \hat{\gamma}$ is a homomorphism of Γ into $\mathrm{Hom}_c(Z, K)$. Finally it is clear that $\hat{\gamma} \equiv e$ iff $\gamma = e$.

3. Since K is central ΓK is a normal closed subgroup of N. The natural map $\lambda : N/\Gamma K \longrightarrow Z$ is a continuous $1-1$ group homomorphism of the Polish group $N/\Gamma K$ onto the compact group Z. By a famous theorem of Souslin λ^{-1} is a Borel isomorphism and by a theorem of Banach the Borel homomorphism λ^{-1} is continuous. Thus λ is a homeomorphism and it now follows easily that also $\lambda^* : N/\Gamma \longrightarrow X$, defined by $\lambda^*(g\Gamma) = gx_0$, $(g \in N)$, is a homeomorphism as well. In particular Γ is co-compact. This completes the proof of Lemma 3.1.
4. Since $\mathrm{Hom}_c(Z, K)$ is isomorphic to $\mathrm{Hom}_c(\hat{K}, \hat{Z})$. It follows that when $\hat{K}$ is finitely generated $\mathrm{Hom}_c(Z, K)$, and therefore also Γ, is a countable group. Since Γ is closed it must be discrete. The rest is now clear. ∎

3.5 Remark: The construction in Lemma 3.4, can be reversed; suppose $\hat{\gamma} \in \mathrm{Hom}_c(Z, K)$ is given then we can define $\gamma \in \mathcal{H}(X)$ by $\gamma(x) = \hat{\gamma}(\pi x)x$. We then have for each $k \in K$

$$\text{(i)} \qquad \gamma(kx) = \hat{\gamma}(\pi kx))kx = \hat{\gamma}(\pi x)kx = k\hat{\gamma}(\pi x)x = k\gamma x$$

and also for $b \in N$, $x \in X$

$$\begin{aligned}\gamma bx &= \hat{\gamma}(\pi bx)bx = \hat{\gamma}(f(b))\hat{\gamma}(\pi x)bx \\ &= \hat{\gamma}(f(b))b\hat{\gamma}(\pi x)x = \hat{\gamma}(f(b))b\gamma x\end{aligned}$$

so that

$$\text{(ii)} \qquad [\gamma, b] = \hat{\gamma}(f(b)) \in K \ .$$

As a result of (i) and (ii) if we let $\tilde{\Gamma}$ be the group of all $\gamma \in \mathcal{H}(X)$ obtained in this way then $\tilde{\Gamma} \supset \Gamma$, $\tilde{\Gamma} \cong \mathrm{Hom}_c(Z, K)$ and the closed group $\tilde{N}$ generated by $\tilde{\Gamma}$ and N is nilpotent of class 2, with $[\tilde{N}, \tilde{N}] \subset K$, and it acts transitively on X. Thus when convenient we can replace N by $\tilde{N}$ and assume that $\Gamma \cong \mathrm{Hom}_c(Z, K)$.

§4. The Enveloping Group of a Nil-Flow of Class 2

In this section we prove Theorem 2.2 which then yields the implication $\hat{1} \Rightarrow \hat{2}$ in Theorem 2.1.

We assume our flow (X, a) satisfies condition $\hat{1}$ of Theorem 2.1.

4.1 PROPOSITION:

1. *Given $p \in E$ define $\varphi_0^p = \lim \varphi_0^{n_k}$, where $\{n_k\}$ is a net in $\mathbb{Z}$ such that $\lim a^{n_k} = p$ in E. Then the limit φ_0^p exists and is independent of the choice of the net $\{n_k\}$.*
2. *The map $p \mapsto \varphi_0^p$ is a homomorphism of (E, a) onto (Φ, φ_0).*
3. *For every $g \in N$ and $p \in E$*

$$pg\Gamma = \varphi(g) gp\Gamma \qquad \text{where} \quad \varphi = \varphi_0^p \ .$$

4. *E acts on $\tilde{E}$ by*

$$p(h\Gamma, \psi) = (ph\Gamma, \varphi_0^p \psi) \qquad ((h\Gamma, \psi) \in \tilde{E},\ p \in E)$$

and the map $j : p \mapsto p(\Gamma, e) = (p\Gamma, \varphi_0^p)$, $j : (E, a) \to (\tilde{E}, \tilde{a})$ where $\tilde{a} = a \times \varphi_0$, is a flow isomorphism.

5. *If $p \in E$ and $j(p) = (p\Gamma, \varphi_0^p) = (h\Gamma, \psi)$ then for every $\gamma \in \Gamma$, $h\Gamma = \varphi(\gamma)\gamma h\Gamma$.*

Proof:

1. Given $g \in N$

$$a^{n_k} g\Gamma = [a^{n_k}, g] g a^{n_k}\Gamma = \varphi_0(g)^{n_k} g a^{n_k}\Gamma.$$

By compactness, the existence of the limits $\lim a^{n_k} g\Gamma = pg\Gamma$ and $\lim g a^{n_k}\Gamma = gp\Gamma$ implies the existence of $\lim \varphi_0(g)^{n_k} = \varphi(g)$, and this limit depends only on p. We also have now $pg\Gamma = \varphi(g)gp\Gamma$ as claimed in 3.

2. This is clear. For 4 we have

$$j(ap) = ap(\Gamma, e) = (ap\Gamma, \varphi_0^{ap}) = (ap\Gamma, \varphi_0\varphi_0^p) = (a \times \varphi_0)(p\Gamma, \varphi_0^p) = \tilde{a}j(p) \ ,$$

so that j is a homomorphism. If $j(p) = j(q)$ then $p\Gamma = q\Gamma$ and $\varphi_0^p = \varphi_0^q$. Hence for every $g \in N$ by 3,

$$pg\Gamma = \varphi_0^p(g)gp\Gamma = \varphi_0^q(g)gq\Gamma = qg\Gamma \ ;$$

this means $p = q$ and j is an isomorphism.

5. This is a special case of 3. When $g = \gamma \in \Gamma$, $h\Gamma = p\Gamma = p\gamma\Gamma = \psi(\gamma)\gamma p\Gamma = \psi(\gamma)\gamma h\Gamma$. ∎

Proof of Theorem 2.2: It is now clear that E is also the enveloping group of the flow $(\tilde{E}, \tilde{a})$ and if for $p, q \in E$, $j(p) = (p\Gamma, \varphi_0^p) = (g\Gamma, \varphi)$ and $j(q) = (q\Gamma, \varphi_0^q) = (h\Gamma, \psi)$ then

$$j(pq) = pj(q) = p(h\Gamma, \psi) = (ph\Gamma, \varphi\psi) = (\varphi(h)hg\Gamma, \varphi\psi) \ .$$

This yields the formula for the product in $\tilde{E}$.

Given $(h\Gamma, \psi) \in \tilde{E}$ there exists $p \in E$ for which $p(h\Gamma, \psi) = (\Gamma, e)$. If $j(p) = (g\Gamma, \varphi)$ then $(g\Gamma, \varphi)(h\Gamma, \psi) = (\varphi(h)hg\Gamma, \varphi\psi) = (\Gamma, e)$ hence $\varphi = \psi^{-1}$ and $\Gamma = \varphi(h)hg\Gamma$. This implies $g\Gamma = \varphi(h)^{-1}h^{-1}\Gamma = \psi(h)h^{-1}\Gamma$ and we get $(h\Gamma, \psi)^{-1} = (\psi(h)h^{-1}\Gamma, \psi^{-1})$ as required. ∎

$\hat{1} \Rightarrow \hat{2}$: We now have an explicit description of the enveloping group E of (X, a) and it is an easy matter to check that it is nilpotent of class 2. First we observe that K is embedded in $\tilde{E}$ as a compact central subgroup:

$$\{(k\Gamma, e) : k \in K\} \ .$$

Next observe that for $(g\Gamma, \varphi), (h\Gamma, \psi) \in \tilde{E}$

$$\begin{aligned}[(g\Gamma, \varphi), (h\Gamma, \psi)] &= (g\Gamma, \varphi)(h\Gamma, \psi)(\varphi(g)g^{-1}\Gamma, \varphi^{-1})(\psi(h)h^{-1}\Gamma, \psi^{-1}) \\ &= (\varphi(h)hg\Gamma, \varphi\psi)(\varphi(g)\psi(h)\varphi^{-1}(h^{-1})h^{-1}g^{-1}\Gamma, \varphi^{-1}\psi^{-1}) \\ &= (\varphi(h)\psi(g^{-1})[h, g]\Gamma, e)\end{aligned}$$

and $[\tilde{E}, \tilde{E}] \subset K$. ∎

§5. Proofs of Implications $\hat{2} \Rightarrow \hat{3} \Rightarrow \hat{4}$ and $\hat{4}^* \Rightarrow \hat{1}$

$\hat{2} \Rightarrow \hat{3}$: The map $\rho : p \mapsto \pi p x_0$ from E onto Z is a group homomorphism and since Z is commutative we have $\rho(r) = \pi r x_0 = \pi x_0 = z_0$ for every $r \in [E,E]$. Thus for such r there exists a $k \in K$ with $rx_0 = kx_0$. Let $p \in E$; then since $[E,E]$ is central $rpx_0 = prx_0 = pkx_0 = kpx_0$. Since $Ex_0 = X$ we conclude that as elements of X^X, $r = k$. Thus $[E,E]$ and therefore also $L =$ closure $[E,E]$, are subgroups of $K \cap E$. L is a compact group of automorphisms of the flow (E,a), and E/L is a factor flow of E. Since E/L is clearly its own enveloping group and since it is commutative we deduce that E/L is almost periodic ([A]). It now follows that $E' \subset L \subset K$. If now $k \in K$ then $\pi x_0 = \pi k x_0$ and therefore there exists $q \in E'$ with $kx_0 = qx_0$. Since $E' \subset K$ this yields $k = q$ and also $K \subset E'$. ∎

$\hat{3} \Rightarrow \hat{4}$: Since $E' = K$ and since the action of K as a subgroup of E on $\Omega \subset X \times X$ is the diagonal action, we conclude that the quotient map $\Omega \xrightarrow{\pi_1} \Omega/\Delta_K = Z_1$ is the largest almost periodic factor of Ω. ∎

Assume now that condition $\hat{4}$ of Theorem 1 holds. Fix $x_1 \in X$ and let $\Omega = \bar{o}(x_0, x_1)$. Put

$$L = \{(k,k') \in K \times K : (k \times k')\Omega = \Omega\},$$
$$K_0 = \{k \in K : (e \times k)\Omega = \Omega\} .$$

By assumption Ω is Δ_K invariant and the quotient map $\Omega \xrightarrow{\pi_1} \Omega/\Delta_K = Z_1$ is the largest almost periodic factor of Ω. It is now clear (identifying k with $e \times k$ $(k \in K_0)$), that K_0 acts freely on Z_1 and that $Z_1/K_0 \cong Z \cong \Omega/L$. Since Z_1 is almost periodic and K_0 is a group of automorphisms, it follows that K_0 can be identified with a subgroup of Z_1, and we have the following short exact sequence of compact abelian metrizable groups

$$1 \to K_0 \to Z_1 \to Z \to 1 .$$

Assume for the moment that this exact sequence splits; i.e. assume $Z_1 = K_0 \oplus Z$. Thus our assumptions now are those of $\hat{4}$ together with the assumption that for every $x_1 \in X$, $Z_1 = K_0 \oplus Z$. We will refer to this as condition $\hat{4}^*$ and will now prove:

$\hat{4}^* \Rightarrow \hat{1}$ Since K_0 is now a factor of Z_1 it follows that K_0 is monothetic and we will let k_0 be the image in K_0 of τ_1 the generator of Z_1. This gives condition (i) of Lemma 3.1. For condition (ii) we put $\theta = \pi_{K_0} \circ \pi_1$ where $\Omega \xrightarrow{\pi_1} \Omega/\Delta_K = Z_1$ is the quotient map and $Z_1 \xrightarrow{\pi_{K_0}} K_0$ is the projection of Z_1 onto K_0. The conditions (iii) are clearly satisfied. Finally we check condition (iv). If $\eta(x,x') = \eta(\bar{x},\bar{x}')$ for $(x,x'), (\bar{x},\bar{x}') \in \Omega$ then $x = \bar{x}$ and $\theta(x,x') = \theta(x,\bar{x}')$. We have $\bar{x}' = kx'$ for some $k \in K_0$ and it follows that $\theta(x,x') = \theta(x,kx') = k\theta(x,x')$, and necessarily $k = e$. This proves that η is 1-1. Given $x \in X$ we choose $x' \in X$ such that $(x,x') \in \Omega$ and then also $(x,kx') \in \Omega$ for each $k \in K_0$. Thus $\eta(x,kx') = (x,\theta(x,kx')) = (x,k\theta(x,x'))$ and we conclude that $\{x\} \times K_0 \subset \eta(\Omega)$. This proves that η is also onto and thus all the conditions of Lemma 3.1 are satisfied. By this lemma (X,a) satisfies 1 and our proof of $\hat{4}^* \Rightarrow \hat{1}$ is complete. ∎

§6. Implication $\hat{4} \Rightarrow \hat{1}$

We now go back to assumption $\hat{4}$ and consider the monothetic compact metrizable group $Z = X/K$. By identifying its dual group $\hat{Z}$ with the set of eigenvalues of the flow (Z,τ), we realize $\hat{Z}$ as a countable subgroup of the circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\} = C$. Put

$$\hat{Z}^* = \{\lambda \in C : \exists n \in \mathbb{Z} \quad \text{such that} \quad \lambda^n \in \hat{Z}\} .$$

Then $\hat{Z}^*$ is a divisible subgroup of C. We consider $\hat{Z}^*$ as a discrete group and let Z^* be its compact monothetic metrizable dual group, with identity element z_0^* and canonical generator (the identity map of $\hat{Z}^*$ into C) τ^*. The inclusion $\hat{Z} \to \hat{Z}^*$ induces a homomorphism (of groups and flows) $(Z^*,\tau^*) \xrightarrow{\zeta} (Z,\tau)$. Denote $W = \ker\zeta$, then as a flow (Z^*,τ^*) is a W-extension of (Z,τ). Since Z is the largest almost periodic factor of X, in the following diagram of minimal flows

$$\begin{array}{ccccc} (X,a) & & & & (Z^*,\tau^*) \\ & {\scriptstyle\pi}\searrow & & \swarrow{\scriptstyle\zeta} & \\ & & (Z,\tau) & & \end{array}$$

X and Z^* are relatively disjoint, i.e. the subset $X^* = \{(x,z) \in X \times Z^* : \pi(x) = \zeta(z)\}$ of $X \times Z^*$ is minimal under $a \times \tau^*$. We denote $a^* = a \times \tau^*$ and $x_0^* = (x_0, z_0^*)$, and let $X^* \xrightarrow{\pi^*} Z^*$ be the projection of X^* on the second coordinate. Then clearly Z^* is the largest almost periodic factor of X^*.

6.1 CLAIM: *Condition $\hat{4}^*$ is satisfied by (X^*, a^*).*

Proof: We identify K with the group $\{k \times e : k \in K\}$ which acts freely on X^* by $(k \times e)(x,z) = (kx, z)$ $((x,z) \in X^*)$. Let (x_1, z_1) be a point in X^* and let $\Omega^* = \bar{o}((x_0, z_0^*),\ (x_1, z_1))$ in $X^* \times X^*$. When (Y,T) is a minimal flow we denote by $Q(Y,T) = Q(Y)$ its regionally proximal relation. A well known theorem states that

$$Q(Y) = R_\kappa = \{(y,y') \in Y \times Y : \kappa(y) = \kappa(y')\}$$

where $Y \xrightarrow{\kappa} Y_1$ is the largest almost periodic factor. Also if $Y \xrightarrow{\lambda} Y_2$ is a homomorphism then $\lambda \times \lambda(Q(Y)) = Q(Y_2)$, (see e.g. [E]).

Now let $((x^*, x^{*\prime}), (\bar{x}^* \bar{x}^{*\prime})) = (((x,z),(x'z')), ((\bar{x},\bar{z}),(\bar{x}', \bar{z}'))) \in Q(\Omega^*)$; then $((x,x'),(\bar{x},\bar{x}')) \in Q(\Omega)$ and by condition $\hat{4}$ on X we have $(\bar{x}, \bar{x}') = (kx, kx')$ for some $k \in K$. Also $((z,z'),(\bar{z},\bar{z}')) \in Q(\pi^* \times \pi^*(\Omega^*))$, and since Z^* is almost periodic we have $\bar{z} = z$ and $\bar{z}' = z'$. Thus $((x^*, x^{*\prime}), (\bar{x}^*, \bar{x}^{*\prime})) = (((x,z),(x',z')), ((kx,z),(kx',z'))) = ((x^*, x^{*\prime}), (kx^*, kx^{*\prime}))$ and $Q(\Omega^*) \subset \bigcup_{k \in K}$ graph $(k \times k)$. Conversely let $k \in K$ and let $((x,z),\ (x',z')) \in \Omega^*$, then by $\hat{4}$ which we assume holds for (X,a), $((x,x'),(kx,kx')) \in Q(\Omega)$. It now follows from the definition of the regionally proximal relation that

$$(((x,z),(x',z')), ((kx,z),(kx',z'))) \in Q(\Omega^*).$$

Thus

$$Q(\Omega^*) = \bigcup_{k \in K} \text{graph } (k \times k)$$

and if we let $\Omega^* \xrightarrow{\pi_1^*} Z_1^*$ be the homomorphism of Ω^* on its largest almost periodic factor then

$$Q(\Omega^*) = R_{\pi_1^*} = \{(w,w') \in \Omega^* \times \Omega^* : \pi_1^*(w) = \pi_1^*(w')\}\ , \qquad Z_1^* \cong \Omega^*/\Delta_K$$

and $\hat{4}$ is satisfied by (X^*, a^*). Since the action of K on X^* is via the X coordinate it is clear that also $K_0 = \{k \in K : (e \times k)\Omega^* = \Omega^*\}$ and that K_0 can be identified with a subgroup of Z_1^* so that $Z_1^*/K_0 \cong Z^*$. Describing this quotient as a short exact sequence and passing to the duals we have

$$1 \longrightarrow K_0 \rightarrow Z_1^* \longrightarrow Z^* \longrightarrow 1$$

and

$$1 \longleftarrow \hat{K}_0 \longleftarrow \hat{Z}_1^* \longleftarrow \hat{Z}^* \longleftarrow 1\ .$$

However $\hat{Z}^*$ being a divisible subgroup of the circle C it is also a divisible subgroup of $\hat{Z}_1^*$ and this sequence splits so that $\hat{Z}_1^* = \hat{Z}^* \oplus \hat{K}_0$ and therefore also $Z_1^* = Z^* \oplus K_0$, proving condition $\hat{4}^*$ for (X^*, a^*). (See e.g. [H-R].) ∎

Since we already have $\hat{4}^* \Rightarrow \hat{1}$ we conclude that (X^*, a^*) is a nil-flow of the form N^*/Γ^* where N^* is a nilpotent subgroup of $\mathcal{H}(X^*)$ acting transitively on X^*, $K \subset N^*$, $a^* \in N^*$, $[N^*, N^*] \subset K$ and Γ^* is the closed co-compact subgroup which fixes $x_0^* = (x_0, z_0^*)$.

We have the commutative diagram

$$\begin{array}{ccccc} & & (X^*, a^*) & & \\ & {\scriptstyle p}\swarrow & & \searrow{\scriptstyle \pi^*} & \\ (X, a) & & & & (Z^*, \tau^*) \\ & {\scriptstyle \pi}\searrow & & \swarrow{\scriptstyle \zeta} & \\ & & (Z, \tau) & & \end{array}$$

where p is the projection of X^* onto its first coordinate. We recall that ζ is a group homomorphism with $W = \ker \zeta$. W acts on (X^*, a^*) as a group of automorphisms where for $w \in W, (x, z) \in X^*$,

$$w(x, z) = (x, wz) .$$

Let $\tilde{N}$ be the subgroup of $\mathcal{H}(X^*)$ generated by W and N^*.

6.2 CLAIM: *$\tilde{N}$ is nilpotent of class 2, K is central in $\tilde{N}$, $[\tilde{N}, \tilde{N}] \subset K$, and denoting $\tilde{\Gamma} = \Gamma^*$ we have $X^* = \tilde{N}/\tilde{\Gamma}$.*

Proof: Let $b \in N^*$ and $w \in W$, then for $h = [w, b]$ we have

$$\begin{aligned} a^* h = a^* w b w^{-1} b^{-1} &= w a^* b w^{-1} b^{-1} = w k b a^* w^{-1} b^{-1} \\ &= k w b w^{-1} k^{-1} b^{-1} a^* = h a^* \end{aligned}$$

where $k = [a^*, b] \in K$. Thus h is an automorphism of (X^*, a^*). Since clearly the action of K and W commute, h induces also an automorphism of (Z^*, τ^*) and being a commutator this latter automorphism is the identity. In particular $h x_0^* = k x_0^*$ for some $k \in K$ and being both automorphisms of X^*, this implies $h = k$. Thus $[w, b] \in K$ and it follows that $[\tilde{N}, \tilde{N}] \subset K$ and that $\tilde{N}$ is nilpotent of class 2. Clearly now $X^* = \tilde{N}/\tilde{\Gamma}$. ∎

Since (X, a) is isomorphic to the flow $(W\backslash\tilde{N}/\tilde{\Gamma}, a^*)$ this completes the proof of Theorem 2.1*. Let $C(W) = \{b \in \tilde{N} : bw = wb,\ \forall w \in W\}$ be the centralizer of W in $\tilde{N}$. Clearly W, K and a are in $C(W)$.

6.3 Claim: *$C(W)$ acts transitively on X^*.*

Proof: For $b \in \tilde{N}$ the map $\varphi_b : g \mapsto [b, g]$ of $\tilde{N}$ into K is a group homomorphism. Let $\chi = \varphi_b \restriction W$, then $\chi \in \mathrm{Hom}_c(W, K)$ and since K is a torus there exists $\hat{\gamma} \in \mathrm{Hom}_c(Z^*, K)$ with $\hat{\gamma} \restriction W = \chi$. As was remarked in Section 3 we can assume that the element $\gamma \in \mathcal{H}(X^*)$, defined by $\gamma x^* = \hat{\gamma}(\pi^*(x^*))x^*$ $(x^* \in X^*)$, is in $\tilde{\Gamma}$. Let $b_1 = b\gamma^{-1}$ then for $w \in W$ and $x^* \in X^*$,

$$w^{-1}b_1wx^* = w^{-1}[b_1, w]wb_1x^* = [b_1, w]b_1x^*.$$

However, $[b_1, w] = [b\gamma^{-1}, w] = [b, w][\gamma^{-1}, w] = \varphi_b(x)\hat{\gamma}^{-1}(w)$.

Now

$$\begin{aligned}\gamma wx^* &= \hat{\gamma}(\pi^*(wx^*))wx^* = \hat{\gamma}(\pi^*w)\hat{\gamma}(\pi^*x^*)wx^* \\ &= \hat{\gamma}(w)w\hat{\gamma}(\pi^*x^*)x^* = \hat{\gamma}(w)w\gamma x^*.\end{aligned}$$

Hence $\hat{\gamma}(w) = [\gamma, w] = \varphi_b(w)$ so that $[b_1, w] = \varphi_b(w)\hat{\gamma}^{-1}(w) = e$ and $b_1 \in C(W)$.

Since $K \subset C(W)$ it is now clear that $C(W)$ acts transitively on X^*. ∎

It follows that (X^*, a^*) is isomorphic to the nil-flow $(C(W)/\tilde{\Gamma} \cap C(W), a^*)$. Let $N = C(W)/W$, $\Gamma = (\tilde{\Gamma} \cap C(W))W/W$ then clearly (X, a) is isomorphic to the nil-flow $(N/\Gamma, a^*)$ and the implication $\hat{4} \Rightarrow \hat{1}$ is proved. ∎

This also completes the proof of Theorem 2.1. ∎

Notice that Claim 6.3 is the only place in our proof of Theorem 2.1 where we used the assumption that K is a torus. The following example will demonstrate the need for passing from the representation $X^* = \tilde{N}/\tilde{\Gamma}$ to the representation $X^* = C(W)/\tilde{\Gamma} \cap C(W)$ in the proof of the implication $\hat{4} \Rightarrow \hat{1}$.

6.4 Example: Let $N = \{(n, z, y) : n \in \mathbb{Z},\ z, y \in \mathbb{T}\}$ with multiplication

$$(n, z, y)\ (n', z', y') = (n + n',\ z + z',\ y + y' + nz').$$

N is a nilpotent group with $[N, N] \subset K$ where $K = \{(0, 0, y) : y \in \mathbb{T}\}$ is its center. Let $a = (2, \alpha, 0)$ where $\alpha \in \mathbb{T}$ is irrational, and let $\Gamma = \{(n, 0, 0) : n \in \mathbb{Z}\}$. The nil-flow $(N/\Gamma, a)$ is isomorphic to the minimal flow $(\mathbb{T}^2, T)$ where

$$T(z, y) = (z + \alpha,\ y + 2z) \qquad ((z, y) \in \mathbb{T}^2).$$

Let now $W = \{(0,0,0),\ (0,\frac{1}{2},0)\} \subset N$. Then W is a compact commutative subgroup of N with $W \cap K = \{e\}$, and for $w = (0,\frac{1}{2},0)$ we have $aw = wa$. Thus W defines a group of automorphisms of $(N/\Gamma, a)$. However the group W is not normalized by Γ, $W\Gamma$ is not a subgroup of N, and the quotient flow $(W\backslash N/\Gamma, a) = (X, a)$ is not isomorphic to the nil-flow $(N/H, a)$ where H is the group generated by W and Γ. However, if we consider the subgroup $C(W) = \{(2n, z, y) : n \in \mathbb{Z},\ z, y \in \mathbb{T}\}$ of N and $C(W) \cap \Gamma = \{(2n, 0, 0) : n \in \mathbb{Z}\}$ of Γ then $(C(W)/C(W) \cap \Gamma, a) \cong (N/\Gamma, a)$ and $(X, a) = (W\backslash N/\Gamma, a)$ is isomorphic to the nil-flow $(C(W)/W/(C(W) \cap \Gamma)W/W, a)$ and by way of the map $\varphi : (2n, z, y) \mapsto (n, 2z, y)$ from $C(W)$ onto N we have:

$$\begin{array}{ccc}
(2n, z, y) & \xrightarrow{a} & (2n+2,\ z+\alpha,\ y+z) \\
\varphi\big\downarrow & & \big\downarrow\varphi \\
(n, 2z, y) & \xrightarrow{b} & (n+1,\ 2z+2\alpha,\ y+2z)
\end{array}$$

where $b = (1, 2\alpha, 0)$. Thus (X, a) is also isomorphic to the nil-flow $(N/\Gamma, b)$ and to the flow $(\mathbb{T}^2, S)$ where $S(z, y) = (z + 2\alpha,\ y + z)$, $\big((z, y) \in \mathbb{T}^2\big)$. ∎

6.5 Example: Again let N and Γ be as in the previous example. Now, however, we take $a = (0, \alpha, \beta)$ where $1, \alpha$ and β are independent over Q. Our flow $(X, a) = (N/\Gamma, a)$ is now isomorphic to the flow $(\mathbb{T}^2, T)$ where $T(z, y) = (z + \alpha,\ y + \beta)$, $\big((z, y) \in \mathbb{T}^2\big)$, and is therefore almost periodic, so that $K = \{e\}$. However $[N, N] = \{(0, 0, y) : y \in \mathbb{T}\} \not\subset K$.

§7. A Proof of Theorem 2.3

Suppose first that $\varphi(z) = nz + \beta$ for $n > 0$ and $\beta \in \mathbb{T}$. Then the flow (X, a) can be represented as a nil-flow in the following way. Take

$$N = \left\{ \begin{pmatrix} 1 & q & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : q \in \mathbb{Z}\ y, z \in \mathbb{T} \right\},$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : q \in \mathbb{Z} \right\}, \qquad a = \begin{pmatrix} 1 & n & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}.$$

Then (X,a) is isomorphic to the nil-flow $(N/\Gamma,a)$.

Conversely let $X=\mathbb{T}\times C$ with $a(z,y)=(z+\alpha,\ y\cdot\varphi(z))$ $(z\in\mathbb{T},\ y\in C)$, where $\alpha\in\mathbb{T}$ is irrational. We assume (X,a) is minimal and that the projection $(X,a)\xrightarrow{\pi}(\mathbb{T},\alpha)$ is the largest almost periodic factor of (X,a). Each $k\in C=K$ defines an automorphism $k(z,y)=(z,yk)$ of (X,a) and we consider $C=K$ as a subgroup of $\mathcal{H}(X)$. Now assume there exists a nilpotent group $N\subset\mathcal{H}(X)$ acting transitively on X with $a\in N$, $K\subset N$, K central in N and $[N,N]\subset K$. We let $\Gamma=\{\gamma\in N:\gamma(0,1)=(0,1)\}$. Then $(N/\Gamma,a)$ is isomorphic to (X,a).

Since each $b\in N$ induces a rotation β on $\mathbb{T}$ we can describe b as a pair (β,u_b) where $u_b:\mathbb{T}\to C$ is continuous and $b(z,y)=(z+\beta,\ u_b(z)y)$. (Incidentally, we have

$$ba(z,y)=b(z+\alpha,\ y\varphi(z))=(z+\alpha+\beta,\ y\varphi(z)u_b(z+\alpha))$$

and

$$ab(z,y)=(z+\alpha+\beta,\ yu_b(z)\varphi(z+\beta))\ .$$

Hence by assumption there exists $\lambda_b\in C$ such that

$$\frac{\varphi(z+\beta)}{\varphi(z)}=\lambda_b\frac{u_b(z+\alpha)}{u_b(z)}\qquad\text{(Lesigne's equation) .)}$$

Also for $b_i=(\beta_i,u_{b_i})\in N\quad(i=1,2)$

$$\begin{aligned}b_2b_1(z,y)&=b_2\left(z+\beta_1,\ yu_{b_1}(z)\right)=(z+\beta_1+\beta_2,\ yu_{b_1}(z)u_{b_2}(z+\beta_1))\\&=(z+\beta_1+\beta_2,\ yu_{b_2b_1}(z))\,.\end{aligned}$$

Hence $u_{b_2b_1}(z)=u_{b_2}(z+\beta_1)u_{b_1}(z)$.

Let D be the subgroup of N consisting of those $d=(\delta,u_d)$ for which the rotation number of u_d is zero. Clearly D is a closed normal subgroup of N containing K. If $d_1(\delta,u_{d_1})$ and $d_2=(\delta,u_{d_2})$ (same δ) then $d_2^{-1}d_1=(0,u^{-1}_{d_2d_1})$ and for some $k\in K$ $d_2^{-1}d_1k(0,1)=(0,1)$ i.e. $d_2^{-1}d_1k\in\Gamma$. However for $\gamma\in\Gamma$ we have $\gamma=(0,u_\gamma)=(0,\hat{\gamma})$ where $\hat{\gamma}:\mathbb{T}\to K$ is a character of $\mathbb{T}$ i.e. $\hat{\gamma}(z)=e^{2\pi inz}$ for some $n\in\mathbb{Z}$. Since $d_2^{-1}d_1k$ is in D it follows that $n=0$ and that $d_2^{-1}d_1k=e$ or $d_2=d_1k$.

Thus the map $f:D\to\mathbb{T}$ given by $f((\delta,u+d))=f(d)=\delta$ is a homomorphism with $\ker f=K$. The map $d\mapsto d(0,0)=(\delta,u_d(0))$ of D into $\mathbb{T}\times K$ is therefore 1-1 and onto (in particular D is compact) and D is isomorphic to $\mathbb{T}\times K$. We can find therefore a subgroup $D_0\subset D$ such that $f:D_0\to\mathbb{T}$ is an isomorphism.

We denote the unique $d \in D_0$ with $f(d) = z$ by $d_z = (z, u_z)$. Define a map $J : \mathbb{T} \times K \to \mathbb{T} \times K$ by $J(z,y) = (z, yu_z(0))$. Then J is a homeomorphism of $\mathbb{T} \times K$ onto itself and $J^{-1}(z,y) = (z, yu_z(0)^{-1})$. Now for some $n \neq 0$ $\deg a = n$ (otherwise $a \in D$ which is compact and our flow would be almost periodic) and for $\gamma = (0, \varphi_n)$ we have $\deg(a\gamma^{-1}) = 0$ i.e. $a\gamma^{-1} \in D$. Thus for some $k_0 \in K$, $a\gamma^{-1}k_0 \in D_0$ and since $f(a\gamma^{-1}k_0) = \alpha$ we must have $a\gamma^{-1}k_0 = (\alpha, u_\alpha)$. On one hand

$$k_0 a\gamma^{-1}(z,y) = (z+\alpha,\ ye^{-2\pi i n z}\varphi(z)k_0)$$

and on the other

$$k_0 a\gamma^{-1}(z,y) = (\alpha, u_\alpha)(z,y) = (z+\alpha,\ yu_\alpha(z)).$$

Therefore $u_\alpha(z) = k_0\varphi(z)e^{-2\pi i n z}$. Now

$$\begin{aligned} J^{-1}aJ(z,y) &= \left(z+\alpha,\ yu_z(0)\varphi(z)u_{z+\alpha}(0)^{-1}\right) \\ &= \left(z+\alpha,\ yu_\alpha(z)^{-1}\varphi(z)\right) \\ &= \left(z+\alpha,\ yk_0^{-1}\varphi(z)^{-1}\varphi(z)e^{2\pi i n z}\right) \\ &= \left(z+\alpha,\ yk_0^{-1}e^{2\pi i n z}\right)\ , \end{aligned}$$

so that denoting $u_z(0) = \psi(z)$ we have

$$\varphi(z) = k_0^{-1}e^{2\pi i n z}\psi(z+\alpha)\psi(z)^{-1} \qquad \blacksquare$$

§8. The General Nil-Flow of Class 2

So far we considered nil-flows of the form $(X,a) = (N/\Gamma, a)$ where $N \subset \mathcal{H}(X)$ is a nilpotent group of class two, for which $[N,N] \subset K$, K a compact group of automorphisms of (X,a) central in N, such that $(Z,\tau) = (X/K, a)$ is the largest almost periodic factor of (X,a). Example 6.5 is an example of a nil-flow of class 2 for which the condition $[N,N] \subset K$ is not satisfied. In this section we will show how the general case can be represented as a nil-flow of class two for a, possibly different, nilpotent group for which the condition $[N,N] \subset K$ does hold (Theorem 2.4). We will then deduce Theorem 2.5. In this section, therefore, our assumptions on the minimal metric flow (X,a) are as follows. There exists a closed subgroup $N \subset \mathcal{H}(X)$ acting transitively on X with $a \in N$, and $[N,N]$ is central in N. We choose $x_0 \in X$ and let

$$\Gamma = \{\gamma \in N : \gamma x_0 = x_0\}\ , \qquad H = \text{closure}\,[N,N]\ .$$

8.1 PROPOSITION:

1. *H is compact.*
2. *There exists a compact subgroup $K \subset H$ such that $(X/K, a) = (Z, \tau)$ is the largest almost periodic factor of (X, a).*

Proof:

1. Let $M =$ closure $H\Gamma$, then μ is a closed subgroup of N centralizing Γ, and the quotient group $\tilde{M} = M/\Gamma$ is compact. The group $\tilde{M}$ acts on (X, a) (on the right) as a group of automorphisms, and the flow $(X/\tilde{M}, a)$, clearly isomorphic to $(N/M, a)$, is almost periodic. We have therefore the following commutative diagram

$$\begin{array}{ccc} (N/\Gamma, a) & & \\ & \searrow^{\pi} & \\ \mu \Big\downarrow & & (Z, \tau) \\ & \swarrow & \\ (N/M, a) & & \end{array}$$

   where (Z, τ) is the largest almost periodic factor of (X, a) and μ is an $\tilde{M}$-extension. Now if $h \in H$ then both h and $\tilde{h}$ its image in $\tilde{M} = M/\Gamma$ are automorphisms of the minimal flow (X, a) and since $hx_0 = \tilde{h}x_0$ we conclude that as elements of $\mathcal{H}(X)$, $h = \tilde{h}$. Since the image of H in $\tilde{M}$ is dense we conclude that H is compact and as a subgroup of $\mathcal{H}(X)$ coincides with $\tilde{M}$.
2. In the commutative diagram above π defines a compact subgroup K of $\tilde{M} = H$ such that $Z = X/K$. ∎

Proof of Theorem 2.4: Statements 1 and 2 are proved in Proposition 8.1. To prove 3 consider the action of $N_1 = N/K$ on $Z = X/K$. This action need not be effective. Let $\Gamma_1 = \{\gamma \in \Gamma : \gamma$ acts as the identity on $Z\}$. Then Γ_1 is a normal subgroup of N and hence $\Gamma_1 K$ is a closed normal subgroup of N_1. Let $N_2 = N/\Gamma_1 K$; then N_2 acts transitively and effectively on Z. Now by assumption τ, the image of a in N_2, acts on Z in an almost periodic way i.e. equicontinuously. Hence the subgroup $T =$ closure $\{\tau^n : n \in \mathbb{Z}\}$ is a compact subgroup of N_2 acting transitively on Z. Let $C(\tau)$ be the centralizes of τ in N_2 and let

$$N_0 = \{g \in N : g\Gamma_1 K \in C(\tau)\}\ .$$

Then $T \subset C(\tau)$, and a and H are in N_0. Clearly N_0 acts transitively on X and if $g_1, g_2 \in N_0$ then $[g_1, g_2] \in \Gamma_1 K \cap [N, N] = K$. Thus $[N_0, N_0] \subset K$ and for $\Gamma_0 = \Gamma \cap N_0$ we have $(N_0/\Gamma_0, a) \cong (X, a)$ as required. ∎

Proof of Theorem 2.5: By Theorem 2.4 we need consider only the case where (X, a) satisfies the equivalent conditions of Theorem 2.1*. Now condition $\hat{2}$ (or $\hat{3}$) of this Theorem is clearly hereditary. ∎

We conclude with some problems which are left open

1. Does Theorem 2.1 hold even without the additional assumption on K?
2. In Theorem 2.1 condition $\hat{4}$, if we require only that Ω is isomorphic to a minimal subset of $X \times Z_1$ where Z_1 is the largest almost periodic factor of Ω (without specifying the nature of $\Omega \xrightarrow{\pi_1} Z_1$), do we still have an equivalent condition?
3. Which parts of this theory can be generalized to nil-flows of class 3, or n?

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